

SECTIONS OF FUNCTIONS AND SOBOLEV TYPE INEQUALITIES

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Dedicated to O.V. Besov on the occasion of his 80th birthday

ABSTRACT. We study functions of two variables whose sections by the lines parallel to the coordinate axis satisfy Lipschitz condition of the order $0 < \alpha \leq 1$. We prove that if for a function f the $\text{Lip } \alpha$ -norms of these sections belong to the Lorentz space $L^{p,1}(\mathbb{R})$ ($p = 1/\alpha$), then f can be modified on a set of measure zero so as to become bounded and uniformly continuous on \mathbb{R}^2 . For $\alpha = 1$ this gives an extension of Sobolev's theorem on continuity of functions of the space $W_1^{2,2}(\mathbb{R}^2)$. We show that the exterior $L^{p,1}$ -norm cannot be replaced by a weaker Lorentz norm $L^{p,q}$ with $q > 1$.

1. INTRODUCTION

The classical embedding with limiting exponent

$$W_1^1(\mathbb{R}^n) \subset L^{n/(n-1)}(\mathbb{R}^n)$$

for the Sobolev space W_1^1 was proved independently by Gagliardo [7] and Nirenberg [11]. Gagliardo's approach was based on estimates of certain mixed norms. A refinement of these estimates and a further development of Gagliardo's method were obtained by Fournier [6]. Different extensions of these results and their applications have been studied, e.g., in the works [1], [2], [5], [8], [9], [10].

In what follows we consider functions of two variables. Let a function f be defined on \mathbb{R}^2 . For any fixed $x \in \mathbb{R}$, the x -section of f (denoted by f_x) is the function of the variable y defined by $f_x(y) = f(x, y)$ ($y \in \mathbb{R}$). Similarly, for a fixed $y \in \mathbb{R}$, the y -section of f is the function $f_y(x) = f(x, y)$ ($x \in \mathbb{R}$) of the variable x . The Gagliardo-Fournier mixed norm space is defined as

$$L_x^1[L_y^\infty] \cap L_y^1[L_x^\infty] \equiv L^1[L^\infty]_{\text{sym}}. \quad (1.1)$$

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Thus, for a function $f \in L^1[L^\infty]_{\text{sym}}$ almost all linear sections are essentially bounded, and the L^∞ -norms of these sections belong to $L^1(\mathbb{R})$. As it was shown in [7], [6], these conditions imply certain integrability properties of f .

The question studied in this paper is a part of a general problem which can be formulated as follows: How do the smoothness conditions imposed on linear sections of a function affect its *global* continuity properties? More precisely, we shall consider mixed norm spaces of functions whose linear sections satisfy Lipschitz conditions.

For any function φ on \mathbb{R} , set $\Delta_h \varphi(t) = \varphi(t+h) - \varphi(t)$. Let $\alpha \in (0, 1]$. Denote by $\text{Lip } \alpha$ the class of all functions $\varphi \in L^\infty(\mathbb{R})$ such that

$$\|\varphi\|_{\text{Lip } \alpha}^* = \sup_{h>0} h^{-\alpha} \|\Delta_h \varphi\|_\infty < \infty.$$

We set also

$$\|\varphi\|_{\text{Lip } \alpha} = \|\varphi\|_\infty + \|\varphi\|_{\text{Lip } \alpha}^*.$$

Some mixed norm spaces of functions with smoothness conditions on sections were studied in the dissertation [1]. In particular, it was proved in [1, Theorem 8.13] that every function

$$f \in L_x^p[(\text{Lip } \alpha)_y] \cap L_y^p[(\text{Lip } \alpha)_x], \quad \text{where } 0 < \alpha \leq 1, 1/\alpha < p < \infty,$$

is equivalent to a bounded and uniformly continuous function on \mathbb{R}^2 . However, the limiting case $p = 1/\alpha$ was left open.

The main objective of the present paper is to study this limiting case. Our interest to this problem is partly motivated by its close relation to embedding of the Sobolev space $W_1^{2,2}(\mathbb{R}^2)$. Note that this relation is similar to the one between embeddings of Gagliardo-Fournier space (1.1) and the Sobolev space $W_1^1(\mathbb{R}^2)$.

Denote by $W_1^{2,2}(\mathbb{R}^2)$ the space of all functions $f \in L^1(\mathbb{R}^2)$ for which pure distributional partial derivatives of the second order $D_1^2 f$ and $D_2^2 f$ exist and belong to $L^1(\mathbb{R}^2)$. It is well known that this doesn't imply the existence of mixed derivatives.

Sobolev's theorem asserts that every function $f \in W_1^{2,2}(\mathbb{R}^2)$ can be modified on a set of measure zero so as to become uniformly continuous and bounded on \mathbb{R}^2 (see [4, Theorems 10.1 and 10.4]).

We have the following embedding

$$W_1^{2,2}(\mathbb{R}^2) \subset L_x^1[(\text{Lip } 1)_y] \cap L_y^1[(\text{Lip } 1)_x] \equiv L^1[\text{Lip } 1]_{\text{sym}} \quad (1.2)$$

(see Proposition 3.2 below).

Let $\alpha \in (0, 1]$. Assume that almost all x -sections and almost all y -sections of f belong to $\text{Lip } \alpha$. We consider the functions

$$\mathcal{N}_\alpha^{(1)} f(y) = \|f_y\|_{\text{Lip } \alpha} \quad \text{and} \quad \mathcal{N}_\alpha^{(2)} f(x) = \|f_x\|_{\text{Lip } \alpha}. \quad (1.3)$$

The integrability properties of these functions provide important characteristics of smoothness of sections. A natural measure of these properties can be obtained in terms of rearrangements of functions (1.3) and their Lorentz norms (cf. [2]).

Denote by $S_0(\mathbb{R}^n)$ the class of all measurable and almost everywhere finite functions f on \mathbb{R}^n such that

$$\lambda_f(y) \equiv |\{x \in \mathbb{R}^n : |f(x)| > y\}| < \infty \quad \text{for each } y > 0.$$

A *non-increasing rearrangement* of a function $f \in S_0(\mathbb{R}^n)$ is a non-negative and non-increasing function f^* on $\mathbb{R}_+ \equiv (0, +\infty)$ which is equimeasurable with $|f|$, that is, for any $y > 0$

$$|\{t \in \mathbb{R}_+ : f^*(t) > y\}| = \lambda_f(y)$$

(see [3, Ch. 1]). The Lorentz space $L^{p,q}(\mathbb{R}^n)$ ($p, q \in [1, \infty)$) is defined as the class of all functions $f \in S_0(\mathbb{R}^n)$ such that

$$\|f\|_{L^{p,q}} \equiv \|f\|_{p,q} = \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} < \infty.$$

We have that $\|f\|_{p,p} = \|f\|_p$. For a fixed p , the Lorentz spaces $L^{p,q}$ strictly increase as the secondary index q increases; that is, the strict embedding $L^{p,q} \subset L^{p,r}$ ($q < r$) holds (see [3, Ch. 4]).

For any $1 \leq p < \infty$, denote

$$\mathcal{U}_p(\mathbb{R}^2) = L_x^{p,1} \left[\left(\text{Lip } \frac{1}{p} \right)_y \right] \cap L_y^{p,1} \left[\left(\text{Lip } \frac{1}{p} \right)_x \right] \equiv L^{p,1} \left[\text{Lip } \frac{1}{p} \right]_{\text{sym}}.$$

For a function $f \in \mathcal{U}_p(\mathbb{R}^2)$, set

$$\|f\|_{\mathcal{U}_p(\mathbb{R}^2)} = \|\mathcal{N}_{1/p}^{(1)} f\|_{p,1} + \|\mathcal{N}_{1/p}^{(2)} f\|_{p,1}.$$

By (1.2), $W_1^{2,2}(\mathbb{R}^2) \subset \mathcal{U}_1(\mathbb{R}^2)$.

The paper is organized as follows. In Section 2 we prove that every function $f \in \mathcal{U}_p(\mathbb{R}^2)$ ($1 \leq p < \infty$) can be modified on a set of measure zero so as to become bounded and uniformly continuous on \mathbb{R}^2 , and we give an estimate of the modulus of continuity of the modified function. This is the main result of the paper. In particular, it provides a generalization of the Sobolev theorem on continuity of functions in $W_1^{2,2}(\mathbb{R}^2)$. We show also that the result is optimal in the sense that the exterior $L^{p,1}$ -norm cannot be replaced by a weaker Lorentz norm $L^{p,q}$ with $q > 1$. In Section 3 we show that the spaces $\mathcal{U}_p(\mathbb{R}^2)$ increase as p increases, and we prove embedding (1.2).

2. CONTINUITY

We recall some definitions and results which will be used in the sequel.

If f is a continuous function on \mathbb{R}^2 , then its modulus of continuity is defined by

$$\omega(f; \delta) = \sup_{0 \leq h, k \leq \delta} |f(x+h, y+k) - f(x, y)|.$$

For any function $f \in S_0(\mathbb{R}^n)$, denote

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(u) du.$$

We shall use the following inequality. Let $g \in S_0(\mathbb{R}^n)$. Then for any $0 < s < t \leq \infty$

$$g^*(s) - g^*(t) \leq \frac{1}{\ln 2} \int_{s/2}^t [g^*(u) - g^*(2u)] \frac{du}{u}. \quad (2.1)$$

Indeed,

$$\begin{aligned} & \int_{s/2}^t [g^*(u) - g^*(2u)] \frac{du}{u} \\ &= \int_{s/2}^s g^*(u) \frac{du}{u} - \int_t^{2t} g^*(u) \frac{du}{u} \geq [g^*(s) - g^*(t)] \ln 2. \end{aligned}$$

It is easy to see that for any $g \in S_0(\mathbb{R}^n)$

$$\lim_{t \rightarrow +\infty} g^*(t) = 0. \quad (2.2)$$

Let $E \subset \mathbb{R}^2$ be a measurable set. For any $y \in \mathbb{R}$, denote by $E(y)$ the y -section of the set E , that is

$$E(y) = \{x \in \mathbb{R} : (x, y) \in E\}.$$

The *essential projection* of E onto the y -axis is defined to be the set Π of all $y \in \mathbb{R}$ such that $E(y)$ is measurable and $\text{mes}_1 E(y) > 0$. Since the function $y \mapsto \text{mes}_1 E(y)$ is measurable, the essential projection is a measurable set in \mathbb{R} . Similarly we define the x -sections and the essential projection of E onto the x -axis.

Theorem 2.1. *Let $1 \leq p < \infty$. Then every function $f \in \mathcal{U}_p(\mathbb{R}^2)$ belongs to $S_0(\mathbb{R}^2)$ and can be modified on a set of measure zero so as to become uniformly continuous and bounded on \mathbb{R}^2 . Moreover, if*

$$\varphi_1(y) = \|f_y\|_{\text{Lip } \frac{1}{p}}^* \quad \text{and} \quad \varphi_2(x) = \|f_x\|_{\text{Lip } \frac{1}{p}}^*,$$

then

$$\|f\|_\infty \leq c(\|\varphi_1\|_{p,1} + \|\varphi_2\|_{p,1}) \quad (2.3)$$

and for the modified function \bar{f} we have that

$$\omega(\bar{f}; \delta) \leq c \int_0^\delta [\varphi_1^*(t) + \varphi_2^*(t)] t^{1/p-1} dt. \quad (2.4)$$

Proof. Let $f \in \mathcal{U}_p(\mathbb{R}^2)$. First we show that $f \in S_0(\mathbb{R}^2)$. Set

$$\psi_1(y) = \|f_y\|_\infty, \quad \psi_2(x) = \|f_x\|_\infty.$$

Then

$$|f(x, y)| \leq \min(\psi_1(y), \psi_2(x)) \quad \text{for almost all } (x, y) \in \mathbb{R}^2. \quad (2.5)$$

It follows that for any $\alpha > 0$ the set $\{(x, y) : |f(x, y)| > \alpha\}$ is contained in the cartesian product $\{x : \psi_2(x) > \alpha\} \times \{y : \psi_1(y) > \alpha\}$, except a subset of measure zero. Thus, $\lambda_f(\alpha) \leq \lambda_{\psi_1}(\alpha) \lambda_{\psi_2}(\alpha)$. Since $\psi_1, \psi_2 \in L^{p,1}(\mathbb{R})$, this implies that $\lambda_f(\alpha) < \infty$ for any $\alpha > 0$.

Now we shall prove that for any $t > 0$

$$f^*(t) - f^*(2t) \leq ct^{1/(2p)} \left(\varphi_1^*\left(\frac{\sqrt{t}}{2}\right) + \varphi_2^*\left(\frac{\sqrt{t}}{2}\right) \right). \quad (2.6)$$

Fix $t > 0$. There exist a set A of type F_σ and a set B of type G_δ such that $A \subset B$, $\text{mes}_2 A = t$, $\text{mes}_2 B = 2t$, and

$$|f(x, y)| \geq f^*(t) \quad \text{for all } (x, y) \in A,$$

$$|f(x, y)| \leq f^*(2t) \quad \text{for all } (x, y) \notin B.$$

At least one of the essential projections of the set A onto the coordinate axes has the one-dimensional measure not smaller than \sqrt{t} . Assume that the projection onto the y -axis has this property, and denote this projection by P . For any $y \in P$, set $\beta(y) = \text{mes}_1 B(y)$. We have

$$\int_P \beta(y) dy \leq \text{mes}_2 B = 2t.$$

This implies that $\text{mes}_1\{y \in P : \beta(y) \geq 4\sqrt{t}\} \leq \sqrt{t}/2$. Hence, there exists a subset $Q \subset P$ of type F_σ such that $\text{mes}_1 Q \geq \sqrt{t}/2$,

$$\beta(y) \leq 4\sqrt{t}, \quad \text{and} \quad f_y \in \text{Lip } \frac{1}{p} \quad (2.7)$$

for any $y \in Q$. Fix $y \in Q$. Observe that the section $A(y)$ has a positive one-dimensional measure. Further, there exists $h \in (0, 8\sqrt{t}]$ such that

$$\text{mes}_1\{x \in A(y) : x + h \notin B(y)\} > 0.$$

Indeed, otherwise for any $h \in (0, 8\sqrt{t}]$ we would have that

$$\chi_{B(y)}(x + h) = 1 \quad \text{for almost all } x \in A(y),$$

where $\chi_{B(y)}$ is the characteristic function of the set $B(y)$. Thus, for any $h \in (0, 8\sqrt{t}]$

$$\int_{A(y)} \chi_{B(y)}(x+h) dx = \text{mes}_1 A(y).$$

Integrating this equality with respect to h and interchanging the order of integrations, we obtain

$$\int_{A(y)} dx \int_0^{8\sqrt{t}} \chi_{B(y)}(x+h) dh = 8\sqrt{t} \text{mes}_1 A(y). \quad (2.8)$$

But $y \in Q$, and therefore, by the first condition in (2.7)

$$\int_0^{8\sqrt{t}} \chi_{B(y)}(x+h) dh \leq \int_{\mathbb{R}} \chi_{B(y)}(u) du = \beta(y) \leq 4\sqrt{t}.$$

This implies that the left-hand side of (2.8) doesn't exceed $4\sqrt{t} \text{mes}_1 A(y)$, and we obtain a contradiction since $\text{mes}_1 A(y) > 0$. Thus, there exists $h \in (0, 8\sqrt{t}]$ such that $|f(x+h, y)| \leq f^*(2t)$ for all x from some subset $A'(y) \subset A(y)$ with $\text{mes}_1 A'(y) > 0$. In addition, $|f(x, y)| \geq f^*(t)$ for all $x \in A(y)$. Thus, we have

$$f^*(t) - f^*(2t) \leq |f(x, y) - f(x+h, y)| \quad \text{for any } x \in A'(y).$$

Since $\text{mes}_1 A'(y) > 0$, this implies that

$$f^*(t) - f^*(2t) \leq \text{ess sup}_{x \in R} |f(x, y) - f(x+h, y)|.$$

Using also the second condition in (2.7), we obtain that

$$f^*(t) - f^*(2t) \leq h^{1/p} \|f_y\|_{\text{Lip } \frac{1}{p}}^* \leq (8\sqrt{t})^{1/p} \varphi_1(y)$$

for any $y \in Q$. Since $\text{mes}_1 Q \geq \sqrt{t}/2$, by the definition of the non-increasing rearrangement we have that $\inf_{y \in Q} \varphi_1(y) \leq \varphi^*(\sqrt{t}/2)$. Hence,

$$f^*(t) - f^*(2t) \leq ct^{1/(2p)} \varphi_1^* \left(\frac{\sqrt{t}}{2} \right).$$

Similarly, in the case when the projection of A onto the x -axis has the one-dimensional measure at least \sqrt{t} , we have the estimate

$$f^*(t) - f^*(2t) \leq ct^{1/(2p)} \varphi_2^* \left(\frac{\sqrt{t}}{2} \right).$$

Thus, we have proved inequality (2.6). Using (2.6), (2.1), and (2.2), we get

$$\begin{aligned} \|f\|_\infty &\leq 2 \int_0^\infty [f^*(t) - f^*(2t)] \frac{dt}{t} \\ &\leq c \int_0^\infty t^{1/(2p)} \left[\varphi_1^* \left(\frac{\sqrt{t}}{2} \right) + \varphi_2^* \left(\frac{\sqrt{t}}{2} \right) \right] \frac{dt}{t} \\ &= c' \int_0^\infty t^{1/p} [\varphi_1^*(t) + \varphi_2^*(t)] \frac{dt}{t}. \end{aligned}$$

This gives (2.3).

Set now

$$f_h(x, y) = \frac{1}{h^2} \int_0^h \int_0^h f(x+u, y+v) du dv \quad (h > 0)$$

and

$$g_h(x, y) = f(x, y) - f_h(x, y).$$

We have (see (2.5))

$$|f_h(x, y)| \leq \min \left(\frac{1}{h} \int_0^h \psi_1(y+v) dv, \frac{1}{h} \int_0^h \psi_2(x+u) du \right).$$

As above, this implies that $f_h \in S_0(\mathbb{R}^2)$ and thus $g_h \in S_0(\mathbb{R}^2)$ for any $h > 0$.

We shall estimate $\|g_h\|_\infty$. First,

$$\begin{aligned} |g_h(x, y)| &\leq \frac{1}{h^2} \int_0^h \int_0^h |f(x+u, y+v) - f(x, y+v)| du dv \\ &\quad + \frac{1}{h} \int_0^h |f(x, y+v) - f(x, y)| dv \\ &\leq h^{1/p} \left[\frac{1}{h} \int_0^h \varphi_1(y+v) dv + \varphi_2(x) \right] \leq h^{1/p} [\varphi_1^{**}(h) + \varphi_2(x)]. \end{aligned}$$

Similarly,

$$|g_h(x, y)| \leq h^{1/p} [\varphi_2^{**}(h) + \varphi_1(y)].$$

There exists a set $E_h \subset \mathbb{R}^2$ of type F_σ such that $\text{mes}_2 E_h \geq h^2$ and

$$|g_h(x, y)| \geq g_h^*(h^2) \quad \text{for all } (x, y) \in E_h.$$

By the estimates obtained above,

$$g_h^*(h^2) \leq h^{1/p} [\varphi_1^{**}(h) + \varphi_2^{**}(h)] + h^{1/p} \min(\varphi_1(y), \varphi_2(x)) \quad (2.9)$$

for any $(x, y) \in E_h$. At least one of the projections of E_h onto the coordinate axes has the one-dimensional measure not smaller than h .

If the projection $\Pi'(E_h)$ onto the x -axis has this property, then

$$\inf_{x \in \Pi'(E_h)} \varphi_2(x) \leq \varphi_2^*(h).$$

Similarly,

$$\inf_{y \in \Pi''(E_h)} \varphi_1(y) \leq \varphi_1^*(h)$$

if $\text{mes}_1 \Pi''(E_h) \geq h$, where $\Pi''(E_h)$ is the projection of E_h onto the y -axis. Thus, using (2.9), we obtain

$$g_h^*(h^2) \leq 2h^{1/p}[\varphi_1^{**}(h) + \varphi_2^{**}(h)]. \quad (2.10)$$

It follows from (2.1) that

$$g_h^*(0+) - g_h^*(h^2) \leq \frac{1}{\ln 2} \int_0^{h^2} [g_h^*(t) - g_h^*(2t)] \frac{dt}{t}. \quad (2.11)$$

Further, we have the following estimates

$$\|f_h(\cdot, y)\|_{\text{Lip } \frac{1}{p}}^* \leq \varphi_1^{**}(h) \quad \text{and} \quad \|f_h(x, \cdot)\|_{\text{Lip } \frac{1}{p}}^* \leq \varphi_2^{**}(h). \quad (2.12)$$

Indeed, for any $\tau > 0$

$$\begin{aligned} & |f_h(x + \tau, y) - f_h(x, y)| \\ & \leq \frac{1}{h^2} \int_0^h \int_0^h |f(x + u + \tau, y + v) - f(x + u, y + v)| du dv \\ & \leq \frac{\tau^{1/p}}{h} \int_0^h \varphi_1(y + v) dv \leq \tau^{1/p} \varphi_1^{**}(h). \end{aligned}$$

This implies the first inequality in (2.12); the second inequality is obtained similarly. Applying (2.12), we get

$$\begin{aligned} \|g_h(\cdot, y)\|_{\text{Lip } \frac{1}{p}}^* & \leq \|f(\cdot, y)\|_{\text{Lip } \frac{1}{p}}^* + \|f_h(\cdot, y)\|_{\text{Lip } \frac{1}{p}}^* \\ & \leq \varphi_1(y) + \varphi_1^{**}(h), \end{aligned}$$

and similarly

$$\|g_h(x, \cdot)\|_{\text{Lip } \frac{1}{p}}^* \leq \varphi_2(x) + \varphi_2^{**}(h).$$

Using these estimates and applying the same reasonings as in the proof of (2.6), we have

$$g_h^*(t) - g_h^*(2t) \leq ct^{1/(2p)} \left[\varphi_1^* \left(\frac{\sqrt{t}}{2} \right) + \varphi_2^* \left(\frac{\sqrt{t}}{2} \right) + \varphi_1^{**}(h) + \varphi_2^{**}(h) \right].$$

This inequality, (2.10), and (2.11) yield that

$$\begin{aligned}
 \|g_h\|_\infty &= g_h^*(0+) \leq 2h^{1/p}[\varphi_1^{**}(h) + \varphi_2^{**}(h)] \\
 &\quad + c[\varphi_1^{**}(h) + \varphi_2^{**}(h)] \int_0^{h^2} t^{1/(2p)-1} dt \\
 &\quad + c \int_0^{h^2} t^{1/(2p)} \left[\varphi_1^* \left(\frac{\sqrt{t}}{2} \right) + \varphi_2^* \left(\frac{\sqrt{t}}{2} \right) \right] \frac{dt}{t} \\
 &\leq c' \left(h^{1/p}[\varphi_1^{**}(h) + \varphi_2^{**}(h)] + \int_0^h t^{1/p}[\varphi_1^*(t) + \varphi_2^*(t)] \frac{dt}{t} \right) \\
 &\leq c'' \int_0^h t^{1/p}[\varphi_1^*(t) + \varphi_2^*(t)] \frac{dt}{t}.
 \end{aligned}$$

It follows that $\|g_h\|_\infty \rightarrow 0$ as $h \rightarrow 0$. Thus, $f_h(x, y)$ converges uniformly on \mathbb{R}^2 as $h \rightarrow 0$, and the limit function \bar{f} is continuous on \mathbb{R}^2 . By the Lebesgue differentiation theorem, $f = \bar{f}$ almost everywhere. Further,

$$\omega(\bar{f}; h) \leq \|g_h\|_\infty + \omega(f_h; h).$$

Applying (2.12), we easily get that

$$\omega(f_h; h) \leq h^{1/p} [\varphi_1^{**}(h) + \varphi_2^{**}(h)].$$

Using this inequality and estimate of $\|g_h\|_\infty$ obtained above, we have

$$\omega(\bar{f}; h) \leq c \int_0^h t^{1/p}[\varphi_1^*(t) + \varphi_2^*(t)] \frac{dt}{t}.$$

The proof is completed. \square

In Theorem 2.1 the exterior $L^{p,1}$ -norm cannot be replaced by a weaker Lorentz norm. More exactly, the following statement holds.

Proposition 2.2. *Let $1 < p < \infty$ and $1 < q < \infty$. Then there exists a function*

$$f \in L^{p,q} \left[\text{Lip } \frac{1}{p} \right]_{\text{sym}} \quad (2.13)$$

such that $f \notin L^\infty(\mathbb{R}^2)$.

Proof. Choose $0 < \beta < 1 - 1/q$. Let

$$g(x, y) = \left| \ln \frac{4}{|x| + |y|} \right|^\beta \quad \text{if } (x, y) \neq (0, 0), \quad g(0, 0) = 0.$$

Further, let $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi(t) = 1$ if $|t| \leq 1/2$, and $\varphi(t) = 0$ if $|t| \geq 1$. Set

$$f(x, y) = g(x, y)\varphi(|x| + |y|).$$

Then $f \notin L^\infty(\mathbb{R}^2)$. We shall prove that

$$f \in L_x^{p,q} \left[\left(\text{Lip } \frac{1}{p} \right)_y \right]. \quad (2.14)$$

Denote for $h \in (0, 1]$

$$\psi(x, h) = \sup_{0 < y \leq 1} h^{-1/p} |g(x, y+h) - g(x, y)| \chi_{(0,1]}(x).$$

It is easy to see that (2.14) will be proved if we show that the function

$$\psi(x) = \sup_{0 < h \leq 1} \psi(x, h) \chi_{(0,1]}(x)$$

belongs to $L^{p,q}(\mathbb{R})$.

Fix $x \in (0, 1]$. Since the function $g(x, y)$ is concave with respect to y on the interval $[0, 2]$, we have that

$$\psi(x, h) = h^{-1/p} \left[\left(\ln \frac{4}{x} \right)^\beta - \left(\ln \frac{4}{x+h} \right)^\beta \right], \quad h \in (0, 1]. \quad (2.15)$$

If $0 < h \leq x$, then

$$\psi(x, h) \leq \frac{4\beta}{x} h^{1-1/p} \left(\ln \frac{2}{x} \right)^{\beta-1} \leq 4\beta x^{-1/p} \left(\ln \frac{2}{x} \right)^{\beta-1}. \quad (2.16)$$

Let now $x \leq h \leq 1$. Set $u = h/x$; then $1 \leq u \leq 1/x$. By (2.15), we have

$$\begin{aligned} \psi(x, h) &\leq (xu)^{-1/p} \left[\left(\ln \frac{4}{x} \right)^\beta - \left(\ln \frac{4}{x(1+u)} \right)^\beta \right] \\ &= (xu)^{-1/p} (z^\beta - (z - \ln(1+u))^\beta) = x^{-1/p} z^\beta u^{-1/p} \left[1 - \left(1 - \frac{\ln(1+u)}{z} \right)^\beta \right], \end{aligned}$$

where

$$z = \ln \frac{4}{x}, \quad z > \ln(1+u).$$

Since

$$1 - (1 - \tau)^\beta \leq c\tau \quad (0 < \beta < 1, 0 < \tau < 1),$$

where c is a constant depending only on β , we obtain

$$u^{-1/p} \left[1 - \left(1 - \frac{\ln(1+u)}{z} \right)^\beta \right] \leq \frac{cu^{-1/p} \ln(1+u)}{z} \leq \frac{c'}{z}.$$

Using this estimate and taking into account (2.16), we have that for all $0 < h \leq 1$

$$\psi(x) = \sup_{0 < h \leq 1} \psi(x, h) \chi_{(0,1]}(x) \leq cx^{-1/p} \left(\ln \frac{2}{x} \right)^{\beta-1}.$$

It follows that $\psi \in L^{p,q}(\mathbb{R})$, and we obtain (2.14). Since $f(x, y) = f(y, x)$, this implies (2.13). \square

3. EMBEDDINGS

For a function $\varphi \in L^\infty(\mathbb{R})$, denote

$$\omega(f; t) = \sup_{0 \leq h \leq t} \|\Delta_h \varphi\|_\infty.$$

If, in addition, $\varphi \in S_0(\mathbb{R})$, then for any $t > 0$

$$\|\varphi\|_\infty \leq \varphi^*(t) + 2\omega(\varphi; t). \quad (3.1)$$

Indeed, for any $\varepsilon > 0$ the set

$$E_\varepsilon = \{x \in \mathbb{R} : |\varphi(x)| > \|\varphi\|_\infty - \varepsilon\}$$

has a positive measure. By the definition of the non-increasing rearrangement, for any $x \in E_\varepsilon$ there exists $h \in (0, 2t)$ such that

$$|\varphi(x + h)| \leq \varphi^*(t).$$

Thus,

$$|\varphi(x)| \leq |\varphi(x) - \varphi(x + h)| + \varphi^*(t) \leq \omega(\varphi; 2t) + \varphi^*(t).$$

This implies (3.1).

Theorem 3.1. *Let $1 \leq p < q < \infty$. Then $\mathcal{U}_p(\mathbb{R}^2) \subset \mathcal{U}_q(\mathbb{R}^2)$. Moreover, for any $f \in \mathcal{U}_p(\mathbb{R}^2)$*

$$\|f\|_{\mathcal{U}_q(\mathbb{R}^2)} \leq c \|f\|_{\mathcal{U}_p(\mathbb{R}^2)}. \quad (3.2)$$

Proof. For any $r \geq 1$, denote

$$\varphi_{r,1}(y) = \sup_{h>0} h^{-1/r} \|\Delta_h f_y\|_\infty \quad \text{and} \quad \varphi_{r,2}(x) = \sup_{h>0} h^{-1/r} \|\Delta_h f_x\|_\infty.$$

We estimate $\varphi_{q,1}^*(t)$. First, let $0 < h \leq t$. Then

$$|f(x + h, y) - f(x, y)| \leq \varphi_{p,1}(y) h^{1/p} \leq \varphi_{p,1}(y) h^{1/q} t^{1/p-1/q}.$$

Thus,

$$\sup_{0 < h \leq t} h^{-1/q} \|\Delta_h f_y\|_\infty \leq \varphi_{p,1}(y) t^{1/p-1/q}. \quad (3.3)$$

In particular, we have that

$$\sup_{0 < h \leq 1} h^{-1/q} \|\Delta_h f_y\|_\infty \leq \varphi_{p,1}(y).$$

On the other hand,

$$\sup_{h \geq 1} h^{-1/q} \|\Delta_h f_y\|_\infty \leq 2 \|f_y\|_\infty.$$

Thus,

$$\varphi_{q,1}(y) \leq 2 \|f_y\|_{\text{Lip } \frac{1}{p}}.$$

Since the function on the right-hand side belongs to $L^{p,1}(\mathbb{R})$, we have that $\varphi_{q,1} \in S_0(\mathbb{R})$.

Let now $h > t$. For any fixed $y \in \mathbb{R}$, we have, applying (3.1)

$$\begin{aligned} \|\Delta_h f_y\|_\infty &\leq (\Delta_h f_y)^*(t) + 2\omega(f_y; t) \\ &\leq (\Delta_h f_y)^*(t) + 2\varphi_{p,1}(y)t^{1/p}. \end{aligned} \quad (3.4)$$

We shall estimate the first term on the right-hand side. For any $\tau > 0$

$$\begin{aligned} \Delta_h f_y(x) &= |f(x+h, y) - f(x, y)| \leq |f(x+h, y) - f(x+h, y+\tau)| \\ &\quad + |f(x, y) - f(x, y+\tau)| + |f(x+h, y+\tau) - f(x, y+\tau)| \\ &\leq [\varphi_{p,2}(x+h) + \varphi_{p,2}(x)] \tau^{1/p} + \varphi_{q,1}(y+\tau)h^{1/q}. \end{aligned}$$

Thus,

$$(\Delta_h f_y)^*(t) \leq 2\varphi_{p,2}^*(t/2)\tau^{1/p} + \varphi_{q,1}(y+\tau)h^{1/q} \quad (\tau > 0).$$

For any fixed y there exists $\tau \in (0, 4t]$ such that $\varphi_{q,1}(y+\tau) \leq \varphi_{q,1}^*(2t)$.

Taking this τ , we obtain

$$(\Delta_h f_y)^*(t) \leq 8\varphi_{p,2}^*(t/2)t^{1/p} + \varphi_{q,1}^*(2t)h^{1/q}.$$

From here and (3.4),

$$\sup_{h \geq t} h^{-1/q} \|\Delta_h f_y\|_\infty \leq 2 [\varphi_{p,1}(y) + 4\varphi_{p,2}^*(t/2)] t^{1/p-1/q} + \varphi_{q,1}^*(2t).$$

This inequality and (3.3) imply that

$$\varphi_{q,1}(y) \leq 4 [\varphi_{p,1}(y) + 2\varphi_{p,2}^*(t/2)] t^{1/p-1/q} + \varphi_{q,1}^*(2t)$$

and therefore

$$\varphi_{q,1}^*(t) - \varphi_{q,1}^*(2t) \leq 4 [\varphi_{p,1}^*(t) + 2\varphi_{p,2}^*(t/2)] t^{1/p-1/q}.$$

Thus,

$$\begin{aligned} &\int_0^\infty [\varphi_{q,1}^*(t) - \varphi_{q,1}^*(2t)] t^{1/q-1} dt \\ &\leq 4 \int_0^\infty \varphi_{p,1}^*(t) t^{1/p-1} dt + 16 \int_0^\infty \varphi_{p,2}^*(t) t^{1/p-1} dt. \end{aligned} \quad (3.5)$$

Since $\varphi_{q,1} \in S_0(\mathbb{R})$, we have by (2.1)

$$\begin{aligned} \int_0^\infty \varphi_{q,1}^*(t) t^{1/q-1} dt &\leq 2 \int_0^\infty t^{1/q-1} \int_{t/2}^\infty [\varphi_{q,1}^*(u) - \varphi_{q,1}^*(2u)] \frac{du}{u} dt \\ &\leq 2^{1+1/q} q \int_0^\infty [\varphi_{q,1}^*(t) - \varphi_{q,1}^*(2t)] t^{1/q-1} dt. \end{aligned}$$

Together with (3.5), this implies that

$$\|\varphi_{q,1}\|_{L^{q,1}} \leq 2^6 q (\|\varphi_{p,1}\|_{L^{p,1}} + \|\varphi_{p,2}\|_{L^{p,1}}).$$

Clearly, a similar estimate holds for $\|\varphi_{q,2}\|_{L^{q,1}}$. Thus, we have

$$\|\varphi_{q,1}\|_{L^{q,1}} + \|\varphi_{q,2}\|_{L^{q,1}} \leq 2^7 q (\|\varphi_{p,1}\|_{L^{p,1}} + \|\varphi_{p,2}\|_{L^{p,1}}). \quad (3.6)$$

Further, let

$$\psi_1(y) = \|f_y\|_\infty, \quad \psi_2(x) = \|f_x\|_\infty.$$

Then

$$\int_1^\infty t^{1/q-1} [\psi_1^*(t) + \psi_2^*(t)] dt \leq \int_1^\infty t^{1/p-1} [\psi_1^*(t) + \psi_2^*(t)] dt$$

and, by (2.3),

$$\int_0^1 t^{1/q-1} [\psi_1^*(t) + \psi_2^*(t)] dt \leq 2q \|f\|_\infty \leq cq (\|\varphi_1\|_{p,1} + \|\varphi_2\|_{p,1}).$$

These estimates together with (3.6) imply (3.2). \square

Finally, we prove embedding (1.2).

Proposition 3.2. *For any function $f \in W_1^{2,2}(\mathbb{R}^2)$*

$$\int_{\mathbb{R}} \|f_x\|_{\text{Lip } 1}^* dx \leq \frac{1}{2} \|D_2^2 f\|_1, \quad \int_{\mathbb{R}} \|f_y\|_{\text{Lip } 1}^* dy \leq \frac{1}{2} \|D_1^2 f\|_1, \quad (3.7)$$

and

$$\|f\|_{u_1(\mathbb{R}^2)} \leq c \|f\|_1^{1/2} (\|D_1^2 f\|_1^{1/2} + \|D_2^2 f\|_1^{1/2}). \quad (3.8)$$

Proof. Let $f \in W_1^{2,2}(\mathbb{R}^2)$. Then by Gagliardo-Nirenberg inequalities (see [7], [11]), the first order weak derivatives $D_1 f$ and $D_2 f$ exist and

$$\|D_1 f\|_1 \leq c \|f\|_1^{1/2} \|D_1^2 f\|_1^{1/2}, \quad \|D_2 f\|_1 \leq c \|f\|_1^{1/2} \|D_2^2 f\|_1^{1/2}. \quad (3.9)$$

For almost all $x \in \mathbb{R}$ we have

$$\|f_x\|_{\text{Lip } 1}^* \leq \|D_2 f(x, \cdot)\|_\infty \leq \frac{1}{2} \int_{\mathbb{R}} |D_2^2 f(x, y)| dy.$$

This implies the first inequality in (3.7); the second inequality follows in the same way.

Further, for almost all $x \in \mathbb{R}$

$$\|f_x\|_\infty \leq \int_{\mathbb{R}} |D_2 f(x, y)| dy.$$

Thus, by (3.9),

$$\int_{\mathbb{R}} \|f_x\|_\infty dx \leq c \|D_2 f\|_1 \leq c \|f\|_1^{1/2} \|D_2^2 f\|_1^{1/2}.$$

Similarly,

$$\int_{\mathbb{R}} \|f_y\|_\infty dy \leq c \|f\|_1^{1/2} \|D_1^2 f\|_1^{1/2}.$$

These estimates together with (3.7) imply (3.8). \square

REFERENCES

- [1] R. Algevrik, *Embedding Theorems for Mixed Norm Spaces and Applications*, PhD Dissertation, Karlstad University Studies, 2010:16.
- [2] R. Algevrik and V.I. Kolyada, *On Fournier-Gagliardo mixed norm spaces*, Ann. Acad. Sci. Fenn. Math. **36** (2011), 493 – 508.
- [3] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston 1988.
- [4] O.V. Besov, V.P. Il'in and S.M. Nikol'skii, *Integral Representations of Functions and Embedding Theorems*, 2nd edition (Russian), Nauka, Moscow, 1996.
- [5] R.C. Blei and J.J.F. Fournier, *Mixed-norm conditions and Lorentz norms*, Commutative harmonic analysis (Canton, NY, 1987), 57 – 78, Contemp. Math., 91, Amer. Math. Soc., Providence, RI, 1989.
- [6] J. Fournier, *Mixed norms and rearrangements: Sobolev's inequality and Littlewood's inequality*, Ann. Mat. Pura Appl. **148** (1987), no. 4, 51 – 76.
- [7] E. Gagliardo, *Proprietà di alcune classi di funzioni in più variabili*, Ricerche Mat. **7** (1958), 102 – 137.
- [8] V.I. Kolyada, *Mixed norms and Sobolev type inequalities*, Banach Center Publ. **72** (2006), 141 – 160.
- [9] V.I. Kolyada, *Iterated rearrangements and Gagliardo-Sobolev type inequalities*, J. Math. Anal. Appl. **387** (2012), 335 – 348.
- [10] M. Milman, *Notes on interpolation of mixed norm spaces and applications*, Quart. J. Math. Oxford Ser. (2) **42** (1991), no. 167, 325 – 334.
- [11] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa **13** (1959), 115 – 162.

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